On identifying codes in lattices

Kenji Obata*

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Abstract

Let $G(V, E)$ be a simple, undirected graph. An *identifying code* on G is a vertex-subset $C \subseteq V$ such that $B(v) \cap C$ is non-empty and distinct for each vertex $v \in V$, where $B(v)$ is a ball about v. Motivated by applications to fault diagnosis in multiprocessor arrays, a number of researchers have considered the problem of constructing identifying codes of minimum density on various two-dimensional lattices, for instance, the square lattice. All currently known constructions of such codes are *periodic*, and none match the best known corresponding lower bounds on the code density. In this work, we give an explicit construction of a two-dimensional lattice G and identifying code C on G such that C is *aperiodic* and has optimal density δ, but every periodic identifying code on G has density greater than δ. In other words, we show that current constructive techniques are, in general, insufficient to achieve optimal code densities.

1 Introduction

Let $G(V, E)$ be a simple, undirected graph, $d(v_1, v_2)$ denote the length of a shortest path between $v_1, v_2 \in V$ in G, and

$$B(v, r) = \{w \in V \ : \ d(v, w) \leq r\}$$

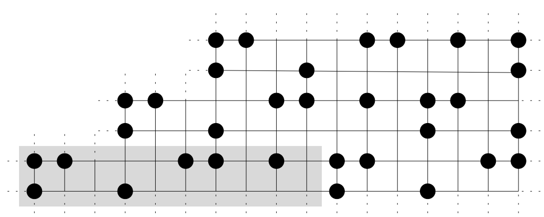
denote the ball of radius r about $v \in V$. A *covering code* of radius r on G is a vertex-subset $C \subseteq V$ such that for every $v \in V$, $B(v, r) \cap C \neq \emptyset$. An *identifying code* of radius r on G is a covering code on G such that, in addition, for every pair of distinct vertices v_1, v_2, $B(v_1, r) \cap C \neq B(v_2, r) \cap C$. If $G'(V, E')$ is the graph formed from G using edge set $E' = \{(v_1, v_2) \in V : d(v_1, v_2) \leq r\}$, then C is an identifying (resp. covering) code of radius r on G iff C is an identifying (resp. covering) code of radius 1 on G'. Thus, without loss of generality, one may assume $r = 1$ and henceforth we write $B(v) = B(v, 1)$.

Identifying codes were introduced by Karpovsky, Chakrabarty, and Levitin [KCL98], who were motivated by the problem of fault diagnosis in multiprocessor arrays. In this application, V represents a set of processors and E a set of links between the processors. One is interested in locating a faulty processor in the array using a subset of diagnostic processors, where each diagnostic processor sends a binary signal indicating whether it or one of its neighbors is faulty. It is immediate that a subset C of diagnostic processors allows up to one faulty processor to be uniquely identified exactly when C is an identifying code on $G(V, E)$. (One could similarly

*kenjioba@cs.berkeley.edu. Computer Science Division, University of California, Berkeley. Work supported by an NSF Graduate Research Fellowship.

define a notion of *list-decodable* identifying codes by requiring only that for every $v \in V$, $|\{w \in V : B(v) \cap C = B(w) \cap C\}| \leq l$ for some list size l.) Clearly, if there exists an identifying code for G then the *complete code* $C = V$ is an identifying code for G. For a given G, however, one would like to find the most *efficient* identifying code on G; that is, to determine an identifying code C with $|C|$ minimum.

With the processor fault detection application in mind, much of the research on identifying codes has focused on determining optimal identifying codes for various two-dimensional lattices, for instance, the square, triangular, or hexagonal lattices in the plane. All currently known constructions of such codes are *periodic*, and none match the best known corresponding lower bounds on the code density. For example, the best identifying code currently known on the square lattice, due to Cohen, Gravier, Honkala, Lobstein, Mollard, Payan, and Zémor [CGH+99], is illustrated below.



This code has density 0.35. On the other hand, it has been shown that any identifying code on the square lattice must have density at least 0.348 [CHLZ99]. The authors conjecture that the code above is optimal. For further examples of known identifying code constructions, the reader is referred to [CHHL01, CHL02].

In this work, we give an explicit construction of a two-dimensional lattice G and identifying code C on G such that C is *aperiodic* and has optimal density δ, but every periodic identifying code on G has density greater than δ. In other words, we show that current constructive techniques are, in general, insufficient to achieve optimal code densities. We leave open the question of whether aperiodicity is required to construct optimal codes on "reasonable" lattices, or if this phenomenon is restricted to structures specialized to this end.

Our construction is based on an aperiodic tiling of the plane with a small set of *Wang tiles*. We encode these tiles using a (traditional) error-correcting code, the *Hadamard code*, and use the result to construct a lattice G such that any optimal covering code on G must be aperiodic. Finally, we convert G into a new lattice G' with the same property over identifying codes.

1.1 Preliminaries

We define a general (two-dimensional) *lattice* as follows. Let

$$F \subseteq \{0, \ldots, w-1\}^2$$

denote a set of edges on the $w \times w$ integer grid. For a subset $P \subseteq \mathbf{Z}^2$ and $a, b \in \mathbf{Z}$, we define

$$P + (a, b) = \{(x + a, y + b) \mid (x, y) \in P\}$$

and

$$E(\mathcal{L}(F)) = \bigcup_{i,j \in \mathbf{Z}} (F + ((w-1)i, (w-1)j)) .$$

It is easy to see that the graph $G(\mathbf{Z}^2, E(\mathcal{L}(F)))$ has an identifying (resp. covering) code iff the subgraph of G induced by the set of non-isolated vertices has an identifying (resp. covering) code. Therefore, without loss of generality, we take the vertex set of $\mathcal{L}(F)$ to be the points in $\mathbf{Z}^2$ which intersect $E(\mathcal{L}(F))$.

We frequently need to refer to a particular "copy" of F in $\mathcal{L}(F)$, for which we write

$$\mathcal{L}(F)_{(i,j)} = \mathcal{L}(F) \cap (\{(w-1)i, \ldots, (w-1)i + w - 1]\} \times \{(w-1)j, \ldots, (w-1)j + w - 1\}) .$$

A code C on G is *periodic* with period (a, b) if there exists a function $g : \mathbf{Z}^2 \to \{0, 1\}$ with

$$g(x, y) = g(x + a, y) = g(x, y + b)$$

for all $x, y \in \mathbf{Z}$ such that

$$C = \{(x, y) : g(x, y) = 1\} \cap G.$$

An identifying code may be viewed as a covering code which satisfies the following additional vertex-pair covering property. This interpretation will be helpful later when moving between the covering and identifying code constructions.

Lemma 1 *C is an identifying code on G iff C is a covering code on G and, for every pair of distinct vertices $v_1, v_2 \in G$ such that $d(v_1, v_2) \leq 2$, there exists $c \in C$ in $B(v_1) \oplus B(v_2)$.*

Proof The condition is plainly necessary. To see that it is sufficient, let v_1, v_2 be any pair of distinct vertices such that $B(v_1) \cap C = B(v_2) \cap C$. If $d(v_1, v_2) \leq 2$ then this is impossible by definition. If $d(v_1, v_2) \geq 3$ then by the covering condition, there exists $c \in C$ such that $v_1 \in B(c)$. Then $d(v_2, c) \geq 2$, so $v_2 \notin B(c)$. Contradiction. ∎

For finite graphs, one is interested in minimizing the order $|C|$ of a code. On lattices, the analogous objective is to minimize the *density* $\delta(C)$ of a code C, where

$$\delta(C) = \limsup_{n \to \infty} \frac{|C \cap \{-n, \ldots, n-1\}^2|}{|\mathcal{L}(F) \cap \{-n, \ldots, n-1\}^2|}.$$

2 Constructions

Our identifying code lattice is constructed in three stages. The base of the construction is an aperiodic *Wang tiling* of the plane; Wang tilings are discussed in section 2.1. In section 2.2, we encode the Wang tile set using a (traditional) error-correcting code, the *Hadamard code*, into a graph F such that every optimal covering code of $\mathcal{L}(F)$ is aperiodic. Finally, in section 2.3, we convert $\mathcal{L}(F)$ into a new lattice $\mathcal{L}(F')$ with the same properties over identifying codes.

2.1 Wang tiles

Let $\{1, \ldots, k\}$ denote a set of k "colors". A *Wang tile* t is a mapping

$$t : \{\leftarrow, \rightarrow, \uparrow, \downarrow\} \rightarrow \{1, \ldots, k\}.$$

For a set $T = \{t_1, \ldots, t_l\}$ of Wang tiles, a *tiling* with T is a function

$$f : \mathbf{Z}^2 \rightarrow \{1, \ldots, l\}$$

such that, for all $i, j \in \mathbf{Z}$ and $d \in \{\uparrow, \rightarrow\}$,

$$t_{f(i,j)}(d) = t_{f((i,j)+v(d))}(d^{-1})$$

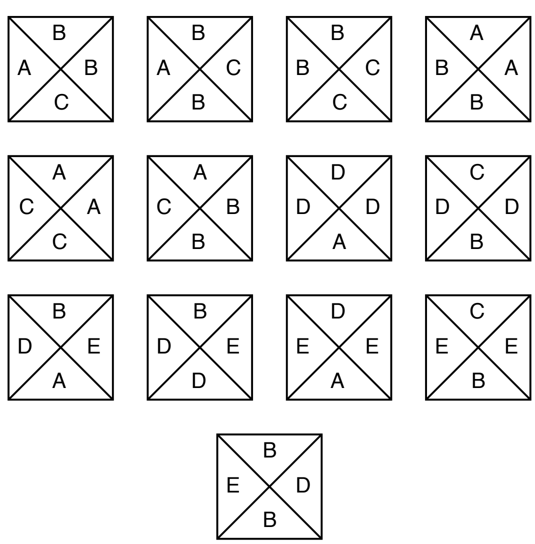
where $d^{-1}(\rightarrow) = \leftarrow, d^{-1}(\uparrow) = \downarrow$ and $v(\rightarrow) = -v(\leftarrow) = (1, 0)$ and $v(\uparrow) = -v(\downarrow) = (0, 1)$. We say that T *admits* a tiling from among a family of functions f if some function in the family is a tiling with T. The function f is called *periodic* with period (a, b) if

$$f(x, y) = f(x + a, y) = f(x, y + b)$$

for every $x, y \in \mathbf{Z}$, and *aperiodic* if no such values (a, b) exist.

Berger [Ber66] constructed a set T of Wang tiles such that T admits a tiling, but does not admit any periodic tiling. Berger's first construction required a set of 20,426 tiles, though this has subsequently been improved by a number of authors. The reader is referred to [GS87] for more details. The most efficient aperiodic Wang tiling currently known, using 13 tiles and 5 colors, is by Culik, using a refinement of an elegant technique due to Kari [Kar96].

Theorem 2 *(Culik [Cul96]) The Wang tile set illustrated below admits an aperiodic tiling (in fact, uncountably many aperiodic tilings), but admits no periodic tiling.*



2.2 Covering code lattice

We define a lattice pattern F on $\{0,\ldots,9\}^2$ using $4\cdot 8+13=45$ vertices as follows. For each $d\in\{\uparrow,\downarrow,\leftarrow,\rightarrow\}$, we have a set B^d of 8 boundary vertices $b_1^d,\ldots,b_8^d$. Vertex $b_i^{\leftarrow}$ is assigned to coordinate $(0,i)$, $b_i^{\uparrow}$ to $(i,9)$, $b_i^{\rightarrow}$ to $(9,i)$, and $b_i^{\downarrow}$ to $(i,0)$. Let $T=\{t_1,\ldots,t_{13}\}$ denote the Wang tile set described in Theorem 2. For $i=1,\ldots,13$ we have a set V of interior vertices v_i, which are assigned to 13 distinct internal coordinates in $\{1,\ldots,8\}^2$ in an arbitrary manner. For every $i,j\in\{1,\ldots,13\}, i\neq j$, we add an edge (v_i,v_j). Finally, for two systems of subsets of $\{1,\ldots,8\}$ $\mathcal{A}^+,\mathcal{A}^-$, to be defined later, we add edges (v_i,b_j^d) for all i,j,d such that $j\in\mathcal{A}_{t_i(d)}^{S(d)}$ where $S(\uparrow)=S(\rightarrow)=+$, $S(\downarrow)=S(\leftarrow)=-$, and $\mathcal{A}_i^{+/-}$ denotes the ith subset in $\mathcal{A}^+,\mathcal{A}^-$, respectively.

It remains to define the systems $\mathcal{A}^{+/-}$. We will use the *Hadamard code*

$$H:\{0,1\}^n\rightarrow\{0,1\}^{2^n},$$

where the $(y_1,\ldots,y_n)$th entry in the codeword of $(x_1,\ldots,x_n)$ is given by

$$H_{y_1,\ldots,y_n}(x_1,\ldots,x_n)=\sum_i x_iy_i,$$

all operations performed over $\mathbf{F}_2$. Let x_1^i,x_2^i,x_3^i denote the binary representation of $i\in\{1,\ldots,5\}$ and y_1^j,y_2^j,y_3^j the binary representation of $j\in\{0,\ldots,7\}$. We set

$$\mathcal{A}_i^+=\{j+1: H_{y_1^j,y_2^j,y_3^j}(x_1^i,x_2^i,x_3^i)=1\}$$

and

$$\mathcal{A}_i^-=\{1,\ldots,8\}\setminus\mathcal{A}_i^+.$$

Theorem 3 *There exists a covering code C on $\mathcal{L}(F)$ with $\delta(C)=1/29$, and this is the best possible density.*

Proof Let f be any tiling with T. Set

$$C=\{(9i+x_{f(i,j)},9j+y_{f(i,j)})\mid(i,j)\in\mathbf{Z}^2\}$$

where (x_k,y_k) denotes the coordinates assigned to the vertex v_k corresponding to tile t_k in F.

For every $(i,j)\in\mathbf{Z}^2$, C contains exactly one vertex v_* from among the 13 vertices $v_1,\ldots,v_{13}$ in $\mathcal{L}(F)_{(i,j)}$. Since all the vertices v_k are connected by edges, every vertex v_k is contained in $B(v_*)$. So we need only verify that every vertex of the form b_i^d is covered by a vertex in C. Consider the boundary vertices $b_1^{\uparrow},\ldots,b_8^{\uparrow}$ in $\mathcal{L}(F)_{(i,j)}$ for any $(i,j)\in\mathbf{Z}^2$. These are synonymous with the vertices ${b'}_1^{\downarrow},\ldots,{b'}_8^{\downarrow}$ in $\mathcal{L}(F)_{(i,j+1)}$. Let $v_{f(i,j)},{v'}_{f(i,j+1)}$ denote the tile vertices selected in $\mathcal{L}(F)_{(i,j)}$ and $\mathcal{L}(F)_{(i,j+1)}$, respectively. By definition of f, $t_{f(i,j)}(\uparrow)=t_{f(i,j+1)}(\downarrow)=c$ for some $c\in\{1,\ldots,5\}$. Since $\mathcal{A}_c^{S(\uparrow)}\cup\mathcal{A}_c^{S(\downarrow)}=\mathcal{A}_c^+\cup\mathcal{A}_c^-=\{1,\ldots,8\}$, every boundary vertex is connected by an edge to one of $v_{f(i,j)},{v'}_{f(i,j+1)}$. An identical argument shows that the same holds for all vertices $b_1^{\rightarrow},\ldots,b_8^{\rightarrow}$.

The asserted density of C is readily verified. The proof of its optimality is deferred until Corollary 9. ■

The primary goal of our construction is the following.

Theorem 4 *Let C be a periodic covering code on $\mathcal{L}(F)$. Then $\delta(C) > 1/29$.*

Before proving Theorem 4, we need a few preliminary lemmas.

Lemma 5 *Let C be a periodic code on $\mathcal{L}(J)$. Then there exist $a, b \in \mathbf{Z}$ such that*

$$C \cap \mathcal{L}(J)_{(i,j)} = [C \cap \mathcal{L}(J)_{(i+a,j)}] - ((w-1)a, 0) = [C \cap \mathcal{L}(J)_{(i,j+b)}] - (0, (w-1)b)$$

for all $(i, j) \in \mathbf{Z}^2$.

Proof If C has period (a', b') then one can take $a = a'$ and $b = b'$. ■

Lemma 6 *Let $\mathcal{A}^+, \mathcal{A}^-$ be as before. Then for any $i_1, i_2 \in \{1, \dots, 5\}$ and $s_1, s_2 \in \{+, -\}$ either $i_1 = i_2$ and $s_1 = -s_2$ so that $\mathcal{A}_{i_1}^{s_1} \cup \mathcal{A}_{i_2}^{s_2} = \{1, \dots, 8\}$, or there exists an index $j \in \{1, \dots, 8\}$ such that $j \notin \mathcal{A}_{i_1}^{s_1} \cup \mathcal{A}_{i_2}^{s_2}$.*

Proof By a standard property of the Hadamard code, the relative Hamming distance between any two codewords or between a codeword and the complement of a different codeword is exactly $\frac{1}{2}$. Further, since we do not encode the message $(0, 0, 0)$ when constructing the sets $\mathcal{A}^{+/-}$, each subset $\mathcal{A}_i^{+/-}$ has cardinality exactly 4. This implies that if $i_1 \neq i_2$,

$$\left|\mathcal{A}_{i_1}^{+/-} \cup \mathcal{A}_{i_2}^{+/-}\right| = 6.$$

If $i_1 = i_2$ then the union has size 4 or 8, when $s_1 = s_2$ and $s_1 = -s_2$, respectively. ■

Lemma 7 *Let C be a covering code on $\mathcal{L}(F)$, and define*

$$w_{(i,j),d}(C) = \frac{1}{8}\left(\left|C \cap \mathcal{L}(V)_{(i,j)}\right| + \left|C \cap \mathcal{L}(V)_{(i,j)+v(d)}\right|\right) + \frac{1}{2}\left|C \cap \mathcal{L}(B^d)_{(i,j)}\right|.$$

Then $w_{(i,j),d} \geq \frac{1}{4}$, and $w_{(i,j),d} = \frac{1}{4}$ exactly when

$$C \cap \mathcal{L}(V)_{(i,j)} = \{v_k\}, C \cap \mathcal{L}(V)_{(i,j)+v(d)} = \{v'_{k'}\}$$

for k, k' such that $t_k(d) = t_{k'}(d^{-1})$, and $\left|C \cap \mathcal{L}(B^d)_{(i,j)}\right| = 0$.

Proof If $C \cap \mathcal{L}(B^d)_{(i,j)}$ is non-empty, then $w_{(i,j),d}(C) \geq \frac{1}{2}$ and we are done. So we may assume that $C \cap \mathcal{L}(B^d)_{(i,j)}$ is empty. The set S of indices s such that the boundary vertex $b_s^d \in \mathcal{L}(B^d)_{(i,j)}$ is adjacent to a vertex in $C \cap \left(\mathcal{L}(V)_{(i,j)} \cup \mathcal{L}(V)_{(i,j)+v(d)}\right)$ is exactly

$$\left(\bigcup_{v_k \in C \cap \mathcal{L}(V)_{(i,j)}} \mathcal{A}_{t_k(d)}^{s(d)}\right) \cup \left(\bigcup_{v_k \in C \cap \mathcal{L}(V)_{(i,j)+v(d)}} \mathcal{A}_{t_k(d^{-1})}^{-s(d)}\right).$$

For all c, $|\mathcal{A}_c^{+/-}| = 4$ so

$$\left|C \cap \left(\mathcal{L}(V)_{(i,j)} \cup \mathcal{L}(V)_{(i,j)+v(d)}\right)\right| \geq 2,$$

which gives $w_{(i,j),d} \geq \frac{1}{4}$. If $w_{(i,j),d} = \frac{1}{4}$ then

$$\left|C \cap \left(\mathcal{L}(V)_{(i,j)} \cup \mathcal{L}(V)_{(i,j)+v(d)}\right)\right| = 2.$$

Further, by Lemma 6, $S = \{1, \dots, 8\}$ exactly when $S = \mathcal{A}_c^+ \cup \mathcal{A}_c^-$ for some $c \in \{1, \dots, 5\}$. ■

Lemma 8 *Let C be a vertex-subset of $\mathcal{L}(F)$ and $a, b \in \mathbf{Z}$. Then*

$$\left| \left| C \cap \left(\cup_{(i,j)} \left(\mathcal{L}(V)_{(i,j)} \cup \mathcal{L}(B^{\leftarrow})_{(i,j)} \cup \mathcal{L}(B^{\downarrow})_{(i,j)} \right) \right) \right| - \sum_{(i,j),d} w_{(i,j),d}(C) \right| \leq 32(a+b),$$

where (i,j) runs through $\{-a, \dots, a-1\} \times \{-b, \dots, b-1\}$, and d through $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$. If a, b are as in Lemma 5, then $32(a+b)$ may be replaced by 0.

Proof The function w weighs vertices in the interior of the summed region exactly according to the number of times they occur in the summation, so these vertices are counted exactly. There are at most $32(a+b)$ vertices remaining along the boundary; when $C \cap \mathcal{L}(F)$ has period a, b, it is easy to check that the boundary vertices are counted exactly as well. ∎

Corollary 9 *The covering code C on $\mathcal{L}(F)$ defined in Theorem 3 is optimal.*

Proof Let C be any covering code on $\mathcal{L}(F)$. Applying Lemma 8, it is easy to see that

$$\delta(C) = \limsup_{a,b \to \infty} \frac{\sum_{(i,j),d} w_{(i,j),d}(C)}{\sum_{(i,j),d} w_{(i,j),d}(C^*)}$$

where C^* is the complete code. By Lemma 7, each term in the numerator is at least $\frac{1}{4}$. By observation, $w_{(i,j),d}(C^*) = \frac{29}{4}$ for all $(i,j), d$. Therefore, $\delta(C) \geq \frac{1/4}{29/4} = 1/29$, as claimed. ∎

It is now straightforward to prove Theorem 4:

Proof Let C be a periodic covering code on $\mathcal{L}(F)$, and $a, b \in \mathbf{Z}$ as in Lemma 5. Then, by the second part of Lemma 8,

$$\delta(C) = \frac{\sum_{(i,j),d} w_{(i,j),d}(C)}{29ab}$$

and each term in the summation is at least $\frac{1}{4}$. If there exist $(i,j), d$ such that $C \cap \mathcal{L}(V)_{(i,j)} = \emptyset$ or $C \cap \mathcal{L}(B^d)_{(i,j)} \neq \emptyset$ then, by Lemma 7, at least one of the $4ab$ terms in the numerator is strictly greater than $\frac{1}{4}$, and so $\delta(C) > 1/29$. Therefore, we may assume that $|C \cap \mathcal{L}(V)_{(i,j)}| = 1$ and $|C \cap \mathcal{L}(B^d)| = 0$ for all $(i,j), d$.

Let $k_{(i,j)}$ denote the index of the unique vertex $v_{k_{(i,j)}} \in C \cap \mathcal{L}(V)_{(i,j)}$. By Theorem 2, the periodic function

$$f(i,j) = k_{(i,j)}$$

cannot be a valid tiling of the aperiodic Wang set T. Therefore, there exist (i,j) and $d \in \{\uparrow, \rightarrow\}$ such that

$$t_{f(i,j)}(d) \neq t_{f((i,j)+v(d))}(d^{-1}).$$

Then, for the tiles $t_{k_{(i,j)}}, t_{k_{(i,j)+v(d)}}$ indexed by the vertices

$$\{v_{k_{(i,j)}}\} = C \cap \mathcal{L}(V)_{(i,j)}, \{v_{k_{(i,j)+v(d)}}\} = C \cap \mathcal{L}(V)_{(i,j)+v(d)},$$

we must have

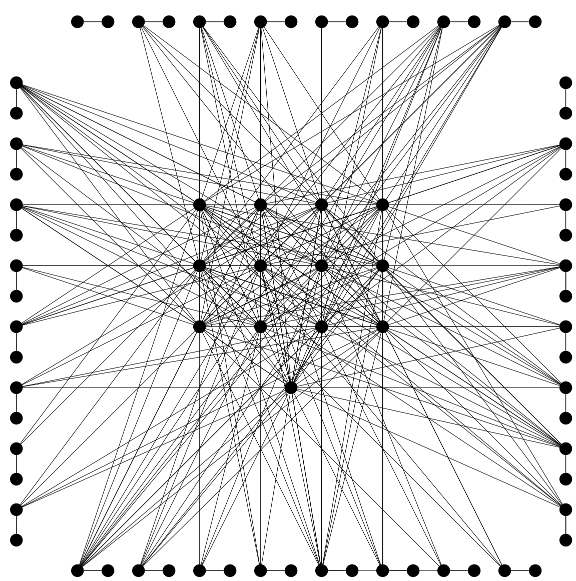
$$t_{k_{(i,j)}}(d) \neq t_{k_{(i,j)+v(d)}}(d^{-1}).$$

By Lemma 7, $w_{(i,j),d}(C) > \frac{1}{4}$, so $\delta(C) > 1/29$. ∎

2.3 Identifying code lattice

A simple modification to our covering code construction yields an identifying code construction with the desired properties.

Let F be as in Section 2.2. We convert $F = V \cup B$ into a $(16+2) \times (16+2)$ lattice pattern $F' = V' \cup B'$ as follows: For each boundary vertex $b_i^d \in B$, we have a corresponding vertex ${b'}_i^d$ in B' along with an additional boundary vertex ${b''}_i^d$ and an edge $({b'}_i^d, {b''}_i^d)$ between them. The vertices ${b'}_i^{\leftarrow} \in B'^{\leftarrow}$ are assigned to coordinate $(0, 2i)$; similarly for the vertices ${b'}_i^{\uparrow}, {b'}_i^{\rightarrow}, {b'}_i^{\downarrow}$. The new vertices ${b''}_i^{\leftarrow}$ are assigned to coordinate $(0, 2i-1)$; similarly for ${b''}_i^{\uparrow}, {b''}_i^{\rightarrow}, {b''}_i^{\downarrow}$. The tile vertices $V \in F$ are copied to V' and assigned distinct coordinates in $\{1, \ldots, 16\}^2$ in an arbitrary manner. The final lattice pattern F' is illustrated below.



Theorem 10 *There exists an identifying code C' on $\mathcal{L}(F')$ such that $\delta(C') = 17/45$, and this is the best possible density.*

Proof We convert the optimal covering code C on $\mathcal{L}(F)$ into an optimal identifying code C' on $\mathcal{L}(F')$ as follows. If $v_k \in C \cap \mathcal{L}(V)_{(i,j)}$ then we place the corresponding vertex ${v'}_k \in C \cap \mathcal{L}(V')_{(i,j)}$ in C'. In addition, we insert all vertices ${b'}_i^d \in B'$ into C'.

Clearly C' is a covering code on $\mathcal{L}(F')$. By Lemma 1, we need only check that for vertices v_1, v_2 with $d(v_1, v_2) \in \{1, 2\}$, there exists $c \in C$ such that $c \in B(v_1) \oplus B(v_2)$. This is confirmed by a straightforward case analysis. The details are omitted from this abstract.

The asserted density of C' is readily verified. The proof of its optimality is deferred until Corollary 12. ■

Lemma 11 *Let C' be an identifying code on $\mathcal{L}(F')$. Then*

(i) For every pair ${b'}_i^d, {b''}_i^d$, $|C' \cap \{{b'}_i^d, {b''}_i^d\}| \geq 1$.

(ii) Every vertex ${b'}_i^d$ is adjacent to a vertex in $C' \cap \mathcal{L}(V')$.

Proof The first claim follows from the requirement that C' cover all of the vertices ${b''}_i^d$. By Lemma 1, for every pair ${b'}_i^d, {b''}_i^d$ there must exist some $c \in C'$ such that $c \in B({b'}_i^d) \oplus B({b''}_i^d)$. The only such vertices in $\mathcal{L}(F')$ are those in $\mathcal{L}(V')$. ■

Corollary 12 *The identifying code C' on $\mathcal{L}(F')$ defined in Theorem 10 is optimal.*

Proof The argument is essentially identical to that for covering codes, with the following additional observations. By the first part of Lemma 11,

$$\left|C' \cap \mathcal{L}({B'}^d)_{(i,j)}\right| \geq 8.$$

By the second part and Lemma 6,

$$\left|C' \cap \left(\mathcal{L}(V')_{(i,j)} \cup \mathcal{L}(V')_{(i,j)+v(d)}\right)\right| \geq 2.$$

Therefore, $w_{(i,j),d}(C') \geq \frac{17}{4}$. Then, as in Corollary 9,

$$\delta(C') = \limsup_{a,b\to\infty} \frac{\sum_{(i,j),d} w_{(i,j),d}(C')}{\sum_{(i,j),d} w_{(i,j),d}(C^*)} \geq \frac{17/4}{45/4} = \frac{17}{45}.$$

■

Theorem 13 *Let C' be a periodic identifying code on $\mathcal{L}(F')$. Then $\delta(C') > 17/45$.*

Proof The argument again follows that for covering codes. As before, equality holds in the expression $w_{(i,j),d} \geq \frac{17}{4}$ exactly when the tiles $t_{k_{(i,j)}}, t_{k_{(i,j)+v(d)}}$ indexed by the vertices

$$\{{v'}_{k_{(i,j)}}\} = C' \cap \mathcal{L}(V')_{(i,j)}, \{{v'}_{k_{(i,j)+v(d)}}\} = C' \cap \mathcal{L}(V')_{(i,j)+v(d)}$$

are such that

$$t_{k_{(i,j)}}(d) = t_{k_{(i,j)+v(d)}}(d^{-1}).$$

By the aperiodicity of T, there exist $(i,j), d$ such that this condition is violated, so at least one term $w_{(i,j),d}(C') > 17/4$ and $\delta(C') > 17/45$. ■

3 Other comments

The method presented here may also be used to show that, given a lattice pattern J such that $\mathcal{L}(J)$ has an optimal periodic covering or identifying code, it is NP-hard to construct an optimal covering or identifying code on $\mathcal{L}(J)$. One simply applies the Wang tile encoding presented here to a standard expression of 3-SAT in terms of tile sets.

Given a set of aperiodic three-dimensional Wang tiles (Wang cubes), one can similarly construct a three-dimensional lattice with the same properties as our two-dimensional constructions. Such a Wang cube set was found by Culik and Kari [CK95] using 21 cubes; in fact, their method generalizes to arbitrary dimensions.

It remains an interesting open problem to determine whether aperiodicity is required to construct optimal codes on "reasonable" lattices, or if this phenomenon is restricted to structures specialized to this end.

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